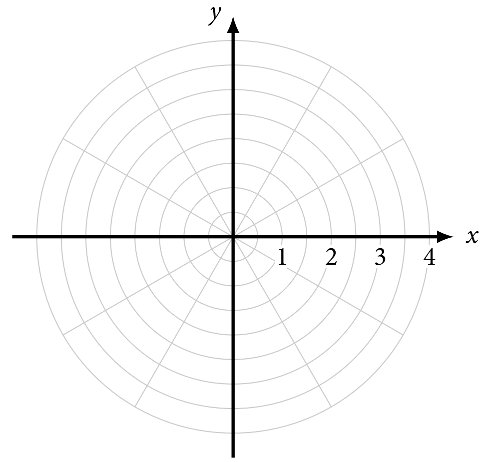
Lesson 24. Double Integrals in Polar Coordinates

1 Review

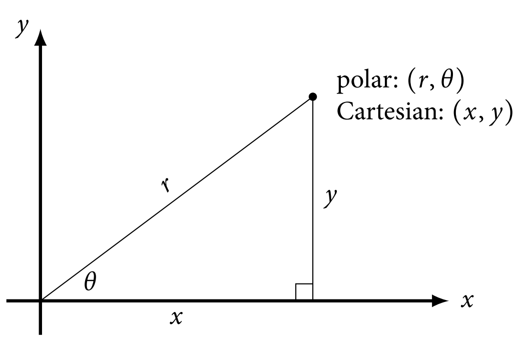
1.1 Polar coordinates

- **Polar coordinate system**: specify points in the xy-plane as (r, θ) where
  - $r =$
  - $\theta =$

Example 1. Sketch the region in the plane consisting of points whose polar coordinates satsify: $1 \leq r \leq 3$, $\pi/6 \leq \theta \leq 5\pi/6$.



1.2 Correspondence between polar and Cartesian coordinates

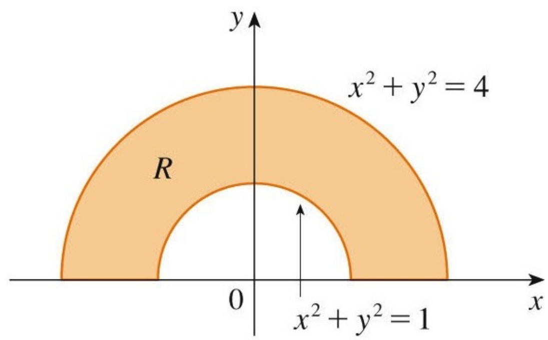


- $x =$
- $y =$
-

Example 2. Find a polar equation for the curve represented by the Cartesian equation $y^2 = 2x - x^2$.

2 Changing to polar coordinates in a double integral

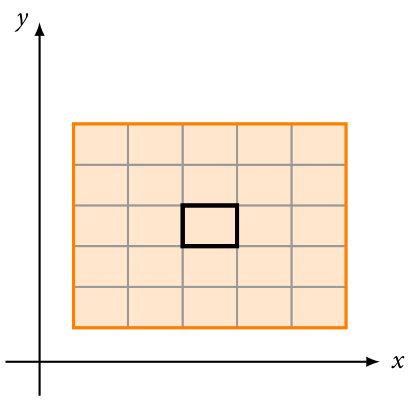
- Idea:
    - Some regions are hard to express in terms of Cartesian coordinates, but easily described using polar coordinates



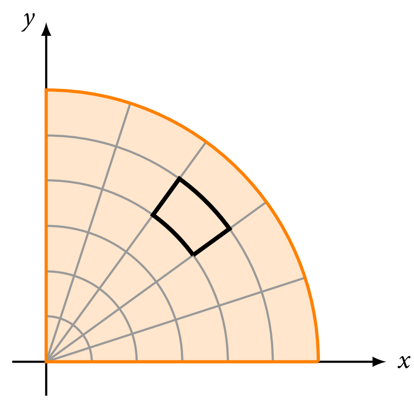
$$R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$$

    - How do we integrate in polar coordinates? Divide regions into **polar subrectangles**

      Cartesian coordinates



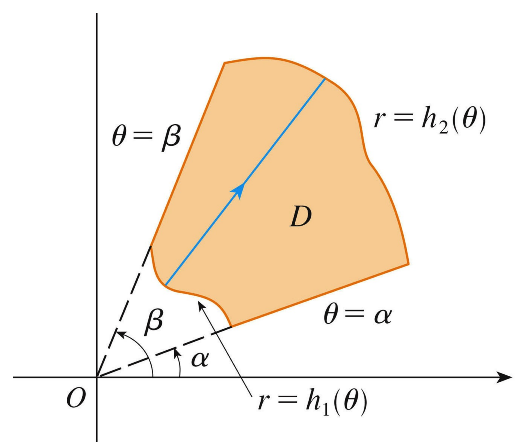
      Polar coordinates



- If D is a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

  then



- Substitute $x = r\cos\theta$ and $y = r\sin\theta$ into $f(x, y)$
- Replace dA with $r\,dr\,d\theta$
- **Don't forget the additional factor r!**

Example 3. Evaluate $\iint_D \sin(x^2 + y^2)\, dA$, where D is the region bounded by the circle $x^2 + y^2 = 4$ and the line $y = 0$.

Example 4. Evaluate $\iint_D (x^2 + y^2)\, dA$, where D is the region in the first quadrant bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.

Example 5. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-x^2-y^2}\, dy\, dx$ by converting to polar coordinates.

Example 6. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.